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Fixed Points on the Real Numbers without the Equality Test

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Abstract

In this paper we present a study of definability properties of fixed points of effective operators on the real numbers without the equality test. In particular we prove that Gandy theorem holds for the reals without the equality test. This provides a useful tool for dealing with recursive definitions using Σ-formulas.

1 Introduction

The aim of the paper is to present a study of definability properties of fixed points of effective operators on the real numbers without the equality test. The question of definability of fixed points of Σ–operators on abstract structures with equality was first studied in [1,3,2]. One of the most fundamental theorems in the area is Gandy theorem which states that the least fixed point of any positive Σ–operator is Σ–definable. This theorem allows us to treat inductive definitions using Σ–formulas. The role of inductive definability as the basic principle of general computability is discussed in [7]. According to general concepts of computable analysis [4,5,11], it is natural to consider languages without equality. Indeed, in all effective approaches to exact real number computation via concrete representations, the equality test is undecidable. This is not surprising, because infinite amount of information must be checked in order to decide that two given numbers are equal.

Until now there has been no Gandy-type theorem known for such structures. Let us note that in all proofs of Gandy theorem that have been known

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so far it is the case that even when the definition of a Σ–operator does not involve equality, the resulting Σ–formula usually does. In this paper we show that it is possible to overcome this problem. In particular we show that Gandy theorem holds for the real numbers without the equality test.

The concept of Σ–definability is closely related to the generalised computability over an abstract structure [1,3,10,12], in particular over the real numbers [8,9,12].

Notions of Σ–definable sets or relations on the real numbers generalise those of computable enumerable sets of natural numbers, and play a leading role in the specification theory that is used in the higher order computation theory over the real numbers. Considering the real numbers without the equality test, we investigate properties of Σ–operators defined on the set of subsets of $\mathbb{R}^n$.

2 Terminology

Throughout the article, we consider the standard model of the real numbers $\langle \mathbb{R}, 0, 1, +, \cdot, -, < \rangle$, denoted also by $\mathbb{R}$, where $+$, $\cdot$ and $-$ are regarded as the usual arithmetic operations on the reals. We use the language of strictly ordered rings, so the predicate $<$ occurs positively in formulas. This allows us to consider Σ-definability as generalisation of computable enumerability. Indeed, in all effective approaches to exact real number computation via concrete representations, we need only finite amount of information in order to show that one given number is less than another one.

3 The Least Fixed Point of Effective Operators

Let us consider the real numbers without the equality test.

In order to do any kind of computation or to develop a computability theory one has to work within a structure rich enough for information to be coded and stored. For this purpose we extend the model $\mathbb{R}$ by the set of hereditarily finite sets $\mathrm{HF}(\mathbb{R})$. Note that such extensions of structures with equality are rather well studied in the theory of admissible sets [1] and used in the theory of abstract state machines [6]. We will construct the set of hereditarily finite sets over the model without equality. This structure permits us to define the natural numbers, and to code and store information via formulas.

We construct the set of hereditarily finite sets, $\mathrm{HF}(\mathbb{R})$, as follows:

(i) $\mathrm{HF}_0(\mathbb{R}) \rightleftharpoons \mathbb{R}$, $\mathrm{HF}_{\mathrm{n+1}}(\mathbb{R}) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_{\mathrm{n}}(\mathbb{R})) \cup \mathrm{HF}_{\mathrm{n}}(\mathbb{R})$, where $n \in \omega$ and for every set B, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of B.

(ii) $\mathrm{HF}(\mathbb{R}) = \bigcup_{n \in \omega} \mathrm{HF}_n(\mathbb{R})$.

We define $\mathbf{HF}(\mathbb{R})$ as the following model:

$$\mathbf{HF}(\mathbb{R}) \rightleftharpoons \langle \mathrm{HF}(\mathbb{R}), \mathbb{R}, \sigma_0, \emptyset, \in, \rangle \rightleftharpoons \langle \mathrm{HF}(\mathbb{R}), \mathbb{R}, \sigma \rangle,$$

where $\sigma_0 = \{0, 1, +, \cdot, -, <\}$, the constant $\emptyset$ stands for the empty set, the binary predicate symbol $\in$ has the set-theoretic interpretation. Let us denote $S(\mathrm{HF}(\mathbb{R})) \rightleftharpoons \mathrm{HF}(\mathbb{R}) \setminus \mathbb{R}$.

The natural numbers $0,\ 1, \dots$ are identified with the (finite) ordinals in $\mathbf{HF}(\mathbb{R})$ i.e. $\emptyset,\ \{\emptyset, \{\emptyset\}\}, \dots$, so in particular, $n+1 = n \cup \{n\}$ and the set ω is a subset of $\mathrm{HF}(\mathbb{R})$.

We use variables subject to the following conventions: $r, r_1, \dots$ range over $\mathbb{R}$ (reals), $x, y, z, s, w, f, g, \dots$ range over $S(\mathrm{HF}(\mathbb{R}))$ (sets), $n, m, l, \dots$ range over ω (natural numbers) and $a, b, c \dots$ range over $\mathrm{HF}(\mathbb{R})$.

The notions of a term and an atomic formula are given in the standard manner.

The set of Δ_0-*formulas* is the closure of the set of atomic formulas under $\wedge, \vee, \neg$, and bounded quantifiers $(\exists a \in s)$ and $(\forall a \in s)$, where $(\exists a \in s)\ \varphi$ denotes $\exists a(a \in s \wedge\ \varphi)$ and $(\forall a \in s)\ \varphi$ denotes $\forall a(a \in s \to \varphi)$.

The set of Σ-*formulas* is the closure of the set of Δ_0 formulas under $\wedge$,$\vee$, $(\exists a \in s)$, $(\forall a \in s)$, and $\exists$.

Recall that the predicate $<$ occurs positively in all Δ_0 and Σ formulas.

Definition 3.1 (i) A set $B \subseteq \mathrm{HF}(\mathbb{R})$ is Σ-*definable*, if there exists a Σ-formula $\Phi(b)$ such that $b \in B \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \Phi(b)$.

(ii) A function $f : \mathrm{HF}(\mathbb{R}) \to \mathrm{HF}(\mathbb{R})$ is Σ-*definable*, if there exists a Σ-formula $\Phi(a, b)$ such that $f(a) = b \leftrightarrow \mathbf{HF}(\mathbb{R}) \models \Phi(a, b)$.

Note that the sets $\mathbb{R}$ and ω are Δ_0–definable. This fact makes $\mathbf{HF}(\mathbb{R})$ a suitable domain for studying subsets of $\mathbb{R}^n$ and operators $\Gamma : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$.

In the following lemma we introduce some Δ_0–definable and Σ–definable predicates that we will use later.

Lemma 3.2 (i) *The predicates $R(a) \rightleftharpoons a \in \mathbb{R}$, $S(a) \rightleftharpoons a$ is a set, and $n \in \omega$ are Δ_0-definable.*

(ii) *The following predicates are Δ_0-definable. $x = y$, $x = y \cap z$,$x = y \cup z$, $x =< y, z >$, $x = y \setminus z$, (recall that all variables x, y, z range over sets)*

(iii) *A function $f : \omega^n \to \omega^m$ is computable if and only if it is Σ-definable.*

(iv) *Let $Fun(g)$ mean that g is a finite function i.e.*

$$g = \{\langle x, y \rangle\, | x, y \text{ range over sets and for every } x \text{ there exists a unique } y\ \}$$

then the predicate $Fun(g)$ is Δ_0-definable.

(v) *If $\mathbf{HF}(\mathbb{R}) \models Fun(g)$ then the domain of g, denoted by δ_g, is Δ_0-definable.*

Proof. Proofs of all properties are straightforward except (iii) which can be found in [3]. □

For finite functions $Fun(f)$ let us denote $f(x) = y$ if $\langle x, y\rangle \in f$.

Proposition 3.3 *(Collection.) For every formula ϕ the following claim holds.*

If $\mathbf{HF}(\mathbb{R}) \models (\forall a \in x)\, \exists b \phi(a, b)$ *then there is a set z such that*

$$\mathbf{HF}(\mathbb{R}) \models (\forall a \in x)\, (\exists b \in z)\, \phi(a, b) \text{ and } \mathbf{HF}(\mathbb{R}) \models (\forall b \in z)\, (\exists a \in x)\, \phi(a, b).$$

Proof. The claim follows from the definition of $\mathbf{HF}(\mathbb{R})$. □

Let $\Phi(x, P)$ be Σ-formula where P occurs positively in Φ.

We think of Φ as defining an operator $\Gamma : \mathcal{P}(\mathrm{HF}(\mathbb{R})) \to \mathcal{P}(\mathrm{HF}(\mathbb{R}))$ given by

$$\Gamma(Q) = \{x | \, (\mathbf{HF}(\mathbb{R}), Q) \models \Phi(x, P)\},$$

where for every set B, $\mathcal{P}(B)$ is the set of all subsets of B. Since the predicate symbol P occurs only positively we have that the corresponding operator Γ is monotone i.e. for any sets from $A \subseteq B$ follows $\Gamma(A) \subseteq \Gamma(B)$.

By monotonicity, the operator Γ has the least (w.r.t. inclusion) fixed point which can be described as follows.

We start from the empty set and apply operator Γ until we reach the fixed point:

$$\Gamma^0 = \emptyset, \quad \Gamma^{\alpha+1} = \Gamma(\Gamma^\alpha), \quad \Gamma^\gamma = \cup_{\beta<\gamma}\Gamma^\beta, \tag{1}$$

where γ is a limit ordinal.

One can easily check that the sets Γ_α form an increasing chain of sets: $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$. By set-theoretical reasons, there exists a minimal ordinal γ such that $\Gamma(\Gamma_\gamma) = \Gamma_\gamma$. This Γ_γ is the least fixed point of the operator Γ.

In order to study the least fixed points of arbitrary Σ–operators (without equality test), we first consider Σ–operators $\Gamma : \mathcal{P}(S(\mathrm{HF}(\mathbb{R}))) \to \mathcal{P}(S(\mathrm{HF}(\mathbb{R})))$. Then we will show how the least fixed points of arbitrary operators can be defined using the least fixed points of such operators. Let us formulate some well–known properties of Σ–operators.

Proposition 3.4 *If Q is a Σ-definable subset of $S(\mathrm{HF}(\mathbb{R}))$ and $x \in \Gamma(Q)$ then there exists $y \in S(\mathrm{HF}(\mathbb{R}))$ such that $y \subseteq Q$ and $x \in \Gamma(y)$.*

Proof. The proof can be found in [3]. □

Proposition 3.5 *The relation $x \in \Gamma(y)$ is Σ-definable.*

Now we are ready to prove Gandy theorem for Σ–operators $\Gamma : \mathcal{P}(S(\mathrm{HF}(\mathbb{R}))) \to \mathcal{P}(S(\mathrm{HF}(\mathbb{R})))$.

Theorem 3.6 *Let $\Gamma : \mathcal{P}(S(\mathrm{HF}(\mathbb{R}))) \to \mathcal{P}(S(\mathrm{HF}(\mathbb{R})))$. Then the least fixed-point of Γ is Σ-definable.*

Proof. We will prove that the least fixed point of the operator Γ is Γ^ω, where Γ^ω is defined as follows: $\Gamma^0 = \emptyset$, $\Gamma^n = \Gamma(\Gamma^{n-1})$ for a finite ordinal n, and $\Gamma^\omega = \bigcup_{m<\omega} \Gamma^m$.

Let us show Σ-definability of Γ^n for every finite ordinal n.

For this purpose we introduce the following family of finite functions:

$$\begin{aligned} X_0 &= <\emptyset, \emptyset>, \\ X_n &= \{f | Fun(f) \text{ and } \delta_f = n+1, f(0) = \emptyset, f \text{ is monotonic} \\ &\qquad \text{and for any } m \le n \text{ the following is true:} f(m) \subseteq \bigcup_{l<m} \Gamma(f(l)\} \end{aligned}$$

where $n > 0$.

From the definitions X_n and Γ it follows that X_n is Σ-definable for all $n \in \omega$, moreover there exists a Σ–formula $\psi(n, x)$ such that $\mathbf{HF}(\mathbb{R}) \models \psi(n, x) \leftrightarrow x \in X_n$.

Below we will use the following useful properties of the families X_n:

(i) Let w be a finite subset of X_n. Define a function $f^*(m) = \cup_{f\in w} f(m)$ for all $m \le n$. Then $f^* \in S(\mathrm{HF}(\mathbb{R}))$ and $f^* \in X_n$.

(ii) If $f \in X_n$ and $m \le n$. Then $f \upharpoonright (m+1) \in X_m$.

(iii) Let $f \in X_m$ and $m \le n$.
Define a function

$$f^*(l) = \begin{cases} f(l), & \text{if } l \le m \\ f(m), & \text{if } m < l \le n. \end{cases}$$

Then $f^* \in X_n$.

(iv) Let $f \in X_n$ and $b \in \Gamma(f(m))$ where $m \le n$.
Define a function

$$f^*(l) = \begin{cases} f(l), & \text{if } l \le n \\ \{b\}, & \text{if } l = n+1. \end{cases}$$

Then $f^* \in X_{n+1}$.

Using these properties let us show that:

$$x \in \Gamma^n \text{ iff } \mathbf{HF}(\mathbb{R}) \models \exists f\,(f \in X_n) \wedge x \in f(n) \tag{2}$$

by induction on n. For $n = 0$ we have $\Gamma^n = \emptyset$ and therefore (2) holds.

Assume that (2) holds for n let us prove that (2) holds for $n+1$.

To prove from left to right let us consider $x \in \Gamma^{n+1} = \Gamma(\Gamma^n)$. By induction hypothesis we have that $x_1 \in \Gamma^n$ iff $\exists g\,(g \in X_n) \wedge x_1 \in g(n)$. So the set Γ^n is Σ-definable. By Proposition 3.4 it follows that there exists $y \in S(\mathrm{HF}(\mathbb{R}))$ such that $y \subseteq \Gamma^n$ and $x \in \Gamma(y)$.

By induction hypothesis and the condition $y \subseteq \Gamma^n$,

$$\mathbf{HF}(\mathbb{R}) \models (\forall z \in y)\, \exists g\, (g \in X_n \wedge z \in g(n))\,.$$

Using Proposition 3.3, we find an element w such that

$$\begin{aligned}\mathbf{HF}(\mathbb{R}) \models\, & (\forall z \in y)\, (\exists g \in w)\, (g \in X_n \wedge z \in g(n)) \wedge \\ & (\forall g \in w)\, (\exists z \in y)\, (g \in X_n \wedge z \in g(n))\,.\end{aligned}$$

Starting from the finite subset $w \subseteq X_n$, we define the function g_0 as follows:

$$g_0(l) = \cup_{g \in w} g(l),\ \ l \leq n.$$

By Property (i) of X_n which is mentioned above, $g_0 \in X_n$. It is easy to check the following inclusion $y \subseteq g_0(n)$. Indeed, if $z \in y$ then there exists $g \in w$ such that $z \in g(n) \subseteq g_0(n)$.
Define a function

$$f(l) = \begin{cases} g_0(l), & \text{if } l \leq n \\ \{x\}, & \text{if } l = n+1. \end{cases}$$

From Property (iv) of X_n follows that $f \in X_{n+1}$ and moreover $x \in f(n+1)$ holds by the definition of f. So f is the required one.

To prove from right to left let us suppose $\exists f\, (f \in X_{n+1}) \wedge x \in f(n+1)$. By the definition of X_{n+1}, $x \in \Gamma(f(m))$ for some $m \leq n$.

Let us check the inclusion : $f(m) \subseteq \Gamma^m$. For this purpose we consider $f_1 = f \restriction (m+1)$. From Property (ii) of X_m follows that $f_1 \in X_m$. So, for all $y \in f_1(m)$ we have $\mathbf{HF}(\mathbb{R}) \models \exists f\, (f \in X_m) \wedge x \in f(m)$. By induction it means that $f_1(m) = f(m) \subseteq \Gamma^m$.

The operator Γ is monotone, so we have

$$x \in \Gamma(f(m)) \subseteq \Gamma(\Gamma^m) \subseteq \bigcup_{m<n+1} \Gamma(\Gamma^m) = \Gamma^{n+1}.$$

Thus we have proven that Γ^n is Σ-definable for all $n \in \omega$. Consequently,

$$x \in \Gamma^\omega \leftrightarrow \exists n \exists f\, (f \in X_n \wedge x \in f(n)) \tag{3}$$

is Σ–definable.

To check that Γ^ω is a fixed point i.e. $\Gamma(\Gamma^\omega) \subseteq \Gamma^\omega$ let us consider $x \in \Gamma(\Gamma^\omega)$ Form 3 it follows that Γ^ω is Σ-definable. From Proposition 3.4 it follows that there exists $y \in S(\mathrm{HF}(\mathbb{R}))$ such that $y \subseteq \Gamma^\omega$ and $x \in \Gamma(y)$. It is easy to check that $y \subseteq \Gamma^m$ for some $m < \omega$. From this we have that $x \in \Gamma(\Gamma^m) \subseteq \Gamma^\omega$ By monotonicity of Γ, the set Γ^ω is the least fixed point. So the least fixed point of the operator Γ is Σ-definable. □

Now we consider $\Phi(r_1, \ldots, r_n, P)$ to be a Σ-formula where P occurs positively in Φ and the arity of P is equal to n.

The formula Φ defines a Σ–operator $\Gamma : \mathcal{P}(\ \mathrm{HF}(\mathbb{R}^n)) \to \mathcal{P}(\mathrm{HF}(\mathbb{R}^n))$ given by

$$\Gamma(Q) = \{(r_1, \ldots, r_n) | \, (\mathbf{HF}(\mathbb{R}), Q) \models \Phi(r_1, \ldots, r_n, P)\}.$$

Theorem 3.7 *Let Γ be an arbitrary Σ–operator $\Gamma : \mathcal{P}(\mathrm{HF}(\mathbb{R})) \to \mathcal{P}(\mathrm{HF}(\mathbb{R}))$. Then the least fixed-point of Γ is Σ-definable.*

Proof.

Without loss of generality let us consider the case $n = 1$.

Let $\Phi(r, P)$ define the operator Γ. We construct a new Σ–operator $F : \mathcal{P}(S(\mathrm{HF}(\mathbb{R}))) \to \mathcal{P}(S(\mathrm{HF}(\mathbb{R})))$ such that

$$r \in \Gamma^n \longleftrightarrow \exists x \, (x \in F^n \wedge r \in x) \, .$$

For this purpose we define the following formula with a new unary predicate symbol Q:

$$\Psi(x, Q) \rightleftharpoons (\forall r \in x) \, \Phi^{P(t)}_{\exists y Q(y) \wedge t \in y}.$$

Define $\Psi(x, Q) = (\forall r \in x) \, (\Phi(r, P))^{P(t)}_{\exists y Q(y) \wedge t \in y}$.

It is easy to see that Ψ induces a Σ–operator F given by

$$F(D) = \{x | (\mathbf{HF}(\mathbb{R}), D) \models \Psi(x, Q)\}.$$

Let us show that

$$r \in \Gamma^n \leftrightarrow \exists x (x \in F^n \wedge r \in x) \tag{4}$$

by induction on n. For $n = 0$ we have $\Gamma^n = F^n = \emptyset$ and therefore (4) holds.

Assume that (4) holds for n let us prove that (4) holds for $n + 1$. In other words we need to prove that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n) \models \Phi(r, P) \leftrightarrow$$
$$(\mathbf{HF}(\mathbb{R}), F^n) \models \exists x \ r \in x \wedge (\forall r' \in x) \, (\Phi(r', P))^{P(t)}_{\exists y Q(y) \wedge t \in y} \, .$$

Since the first formula does not contain Q and the second formula does not contain P it is sufficient to consider one structure $(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n)$ and prove that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \Phi(r, P) \leftrightarrow$$
$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \exists x \ r \in x \wedge (\forall r' \in x) \, (\Phi(r', P))^{P(t)}_{\exists y Q(y) \wedge t \in y} \, .$$

To prove from left to right let us consider $r \in \mathbf{HF}(\mathbb{R})$ such that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \Phi(r, P).$$

Consider the formula $(\Phi(r, P))^{P(t)}_{\exists y Q(y) \wedge t \in y}$ then by induction hypothesis we have that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \forall r' \, (P(r') \leftrightarrow \exists x (x \in Q \wedge r' \in x)) \tag{5}$$

and therefore (by replacement lemma) we have

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models (\Phi(r, P))^{P(t)}_{\exists y Q(y) \wedge t \in y} \, .$$

Now it is easy to check that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \exists x \; r \in x \wedge (\forall r' \in x) \, (\Phi(r', P))^{P(t)}_{\exists y Q(y) \wedge t \in y}$$

taking $x = \{r\}$.

To prove from right to left let us consider $r \in \mathbf{HF}(\mathbb{R})$ such that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \exists x \;\; r \in x \wedge (\forall r' \in x) \, (\Phi(r', P))^{P(t)}_{\exists y Q(y) \wedge t \in y} .$$

From this we have that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models (\Phi(r, P))^{P(t)}_{\exists y Q(y) \wedge t \in y}$$

and form (5) (by replacement lemma) we obtain that

$$(\mathbf{HF}(\mathbb{R}), \Gamma^n, F^n) \models \Phi(r, P).$$

Now from Theorem 3.6 it follows that the least fixed point of the operator F is Σ–definable and therefore the the least fixed point of the operator Γ is also Σ–definable.

$\square$

4 Future work

One of the applications of Gandy theorem in the case of structures with equality is that it allows us to define universal Σ–predicates. It leads to a topological characterisation of Σ–relations on $\mathbb{R}$. Thus the sets $B \subseteq \mathbb{R}^n$ that are Σ-definable in $\mathbf{HF}(\mathbb{R})$ with equality are exactly the effective unions of semialgebraic sets.

We think that Gandy theorem can be used in this way for the structures without equality, but for this we need more evolved arguments. Also we think that it is possible to show that the sets $B \subseteq \mathbb{R}^n$ that are Σ–definable in $\mathbf{HF}(\mathbb{R})$ without equality are exactly the effective unions of open semialgebraic sets.

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